FIBER ORIENTATION AND EJECTION FRACTION IN THE HUMAN LEFT VENTRICLE

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ABSTRACT Fiber orientation and ventricular geometry are incorporated in a mathematical model for ejection fraction of the human left ventricle. The inadequacy of circularly oriented or "constrictor" fibers to explain high ejection fractions of cylindrical, spherical, or ellipsoidal reference ventricles is demonstrated. A class of "helical" fibers is then introduced from which ejection fractions predicted by physiologic amounts of fiber shortening on cylindrical and ellipsoidal reference ventricles are shown to be consistent with those calculated from biplane angiocardiographic films.

INTRODUCTION

In any attempt to relate left ventricular geometry, fiber orientation and ejection fraction, two important physiologic findings must be considered. First, ejection fractions in excess of 0.5 for the normal human left ventricle are commonly determined from biplane angiocardiograms (1) and second, preparations of isolated papillary muscle rarely exhibit shortening upon stimulation in excess of 20% (2—5). That a model for fiber orientation which is consistent with these findings is not trivial is demonstrated by the following result which is proved later: circularly oriented, or "constrictor" fibers forming the surface of a cylinder, sphere, or ellipse of revolution can account for an ejection fraction of at most 0.488 in the spherical case and 0.36 in the cylindrical or ellipsoidal cases even permitting 20% shortening of individual fibers! Anatomical evidence (6), however, indicates that while many fibers are constrictor, or nearly so, those closer to the endocardial (and epicardial) surface of the left ventricle are definitely "helical". A class of such "helical" fibers will be defined and it will be shown that significant ejection fractions can be achieved for such fibers forming a cylindrical or ellipsoidal surface while undergoing only physiologic amounts of shortening.

The geometries considered for the endocardial, or bounding, surface of the left ventricle together with the parameters used to describe an arbitrary point P on each of these surfaces are shown in Fig. 1, where for convenience the modelled ventricle has been inverted so that the apex is pointing upward.

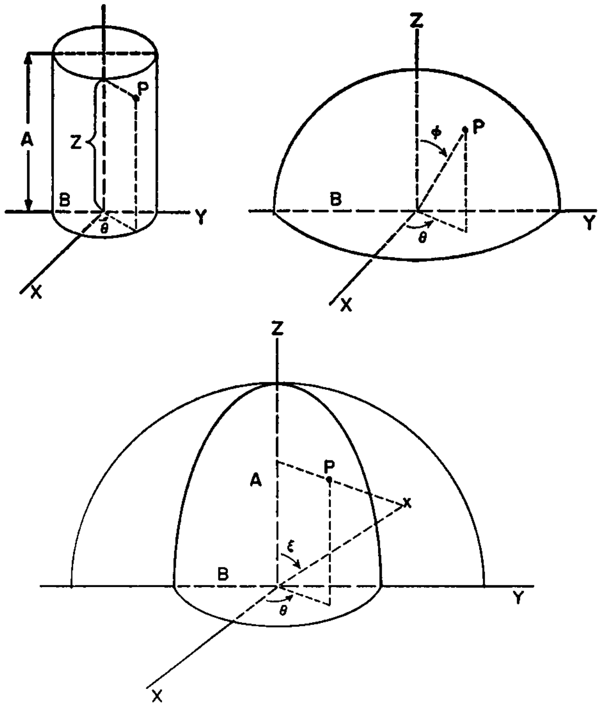


FIGURE 1 Assumed geometric models for the endocardial surface of the human left ventricle; clockwise from upper left the cylinder, the hemisphere and the hemi-ellipse of revolution. The coordinates of an arbitrary point P depend only upon two parameters: θ and z for the cylinder; θ and φ for the hemisphere; θ and ξ for the hemi-ellipse of revolution.

As indicated in the figure, we will only use the hemisphere, hemiellipsoid, or cylinder. This is done for simplicity of presentation; the concepts developed are equally applicable to any other fraction of the total surface provided the resulting surface still remains a surface of revolution.

CIRCULAR OR CONSTRICTOR FIBERS

Fig. 2 illustrates the simple case in which the total surface is assumed to be covered by circular fibers. If each fiber were to shorten concentrically from its initial length, S, to some fraction $k(0 < k < 1)$, of its original length, the volume would decrease from its initial or end diastolic volume V_I to its final or end systolic volume V_F. The ejection fraction, EF, defined as $(V_I - V_F)/V_I$ may be readily calculated. For a hemisphere of initial radius B, for example, $V_I = 2\pi B^3/3$, $V_F = 2\pi k^3 B^3/3$ and $EF = 1 - k^3$. Similar calculations yield an ejection fraction of $1 - k^2$ for both the cylinder and the ellipse of revolution.

The implication of these findings is that large ejection fractions are not compatible

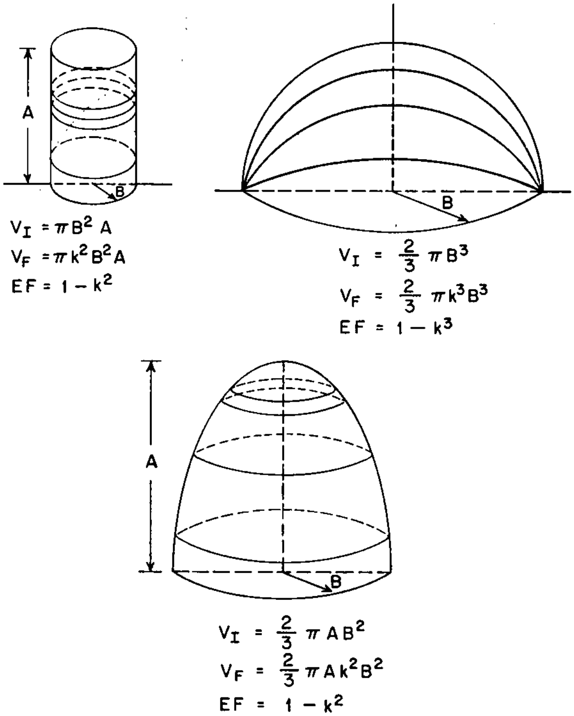


FIGURE 2 Schematic indicating circular fibers forming cylindrical, spherical, and ellipsoidal surfaces. Upon stimulation, uniform shortening of the individual fibers to k times their initial lengths results in a radius change from B to kB with the indicated ejection fractions, EF.

with physiologic amounts of shortening in the case where the fibers form circles. Indeed, $k = 0.8$, which corresponds to a 20% shortening, corresponds to an ejection fraction of only 0.36 for the ellipse of revolution or cylinder and 0.488 for the sphere in contrast to normal ejection fractions in excess of 0.5 determined by biplane angiocardiography (1).

HELICAL FIBERS

Introduction

As illustrated in Fig. 1, the surfaces to be considered are described parametrically by θ and z (the cylinder), θ and φ (the sphere) and by θ and ξ (the ellipse of revolution). If the two parameters defining a surface are functions of a new variable t, a curve is defined which lies wholly on that surface. The following simple functions of t generate a class of curves which are "helical" and which we conceive of as representing helically wound fibers. Only this simple family of curves will be considered.

$$\theta = \Omega t, \; z = A(1 - t), \qquad 0 \leqq t \leqq 1 \qquad \text{Cylinder} \tag{1 a}$$

$$\theta = \Omega t, \; \varphi = t, \qquad 0 \leqq t \leqq \pi/2 \qquad \text{Sphere} \tag{1 b}$$

$$\theta = \Omega t, \; \xi = t, \qquad 0 \leqq t \leqq \pi/2 \qquad \text{Ellipse of revolution.} \tag{1 c}$$

Fig. 3 schematically represents the helical nature of such curves. θ_0, which represents some arbitrary initial angle, may be assumed to be zero due to the rotational symmetry of the surfaces. Ω will be referred to in the following as the helix parameter or as the fiber parameter.

Cylinder

Assume (see Fig. 3) that a helical fiber with parameter Ω lies on the surface of a cylinder of height A and radius B. Substituting the appropriate fiber equation (1 *a*) into the parametric equations of the surface, the initial fiber length S can be shown to be

$$S = [\Omega^2 B^2 + A^2]^{1/2}.$$

If A is held fixed and the fiber allowed to shorten from length S to kS, $0 < k < 1$, the radius B decreases to $\hat{B}$, where

$$[\Omega^2 \hat{B}^2 + A^2]^{1/2} = kS = k[\Omega^2 B^2 + A^2]^{1/2}. \tag{2}$$

The ejection fraction, EF, given by

$$EF = 1 - (\hat{B}/B)^2,$$

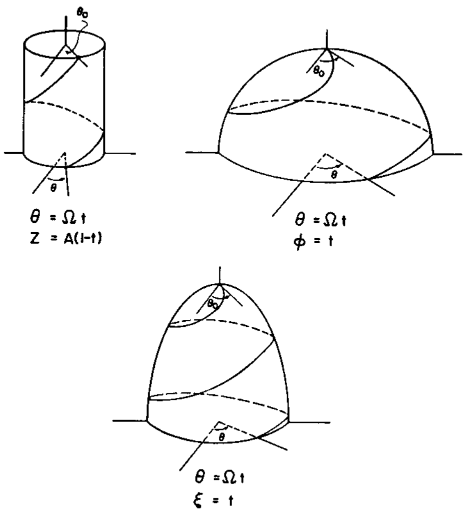


FIGURE 3 A schematic representation of the modelled helical fibers. The parametric equations for the fiber are given below the corresponding figure. The starting angle, θ_0, is arbitrary due to the rotational symmetry of each figure.

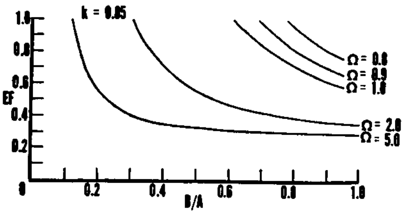


FIGURE 4 Ejection vs. geometry: fixed fiber shortening—cylinder with fixed major axis. Fiber shortening has arbitrarily been set at 15% ($k = 0.85$) and ejection fraction plotted for several values of the fiber parameter, $\Omega = 0.8, 0.9, 1.0, 2.0$, and 5.0.

becomes upon substituting for $\hat{B}$ from equation 2

$$EF = (1 - k^2)\left[1 + \frac{1}{\Omega^2}\left(\frac{A}{B}\right)^2\right]$$

subject to the constraint that $kS \geq A$ (the new fiber length can be no shorter than the cylinder height).

Fig. 4 illustrates the special case of 15% fiber shortening, i.e., $k = 0.85$. Ejection fraction is clearly shown to be a decreasing function of both Ω and B/A for any fixed value of k.

A more realistic model should permit shortening of the cylinder (corresponding to apex-to-base shortening during systole) simultaneously with changes in radius. Let q be the ratio of final cylinder height to initial cylinder height upon shortening of the fibers from length S to kS. The ejection fraction then depends upon q as well as k, A/B, and Ω:

$$EF = 1 - qk^2 + q(q^2 - k^2)(A/B)^2(1/\Omega)^2$$

provided $kS \geq qA$ (the new fiber length can be no shorter than the new cylinder height). For $q > k$, that is for less cylinder shortening than fiber shortening, the ejection fraction may be made arbitrarily close to 1.0 by proper choice of Ω. Indeed if $\Omega^2 = (A/B)^2(q^2 - k^2)/k^2$, $EF = 1.0$. If $q = k$, the ejection fraction becomes $1 - k^3$, as in the case of circular fibers on a sphere, and is thus independent of the fiber parameter Ω and cannot become arbitrarily close to 1.0 except under the physiologically unacceptable condition that k be extremely small.

Sphere

The length, S, of a helical fiber with parameter Ω wrapped on a hemisphere of radius B is given by

$$S = B\,\delta\,E(K, \pi/2)$$

where $K^2 = \Omega^2/(1 + \Omega^2)$, $\delta^2 = 1 + \Omega^2$ and $E(K, \pi/2)$ is the complete elliptic integral of the second kind

$$E(K, \pi/2) = \int_0^{\pi/2} [1 - K^2 \sin^2 y]^{1/2}\, dy.$$

Thus, if the fiber shortens from S to kS, $0 < k < 1$, the radius B will decrease to kB (since δ and E are independent of B). The new volume is $\frac{2}{3}\pi k^3B^3$, the original $\frac{2}{3}\pi B^3$, and the ejection fraction

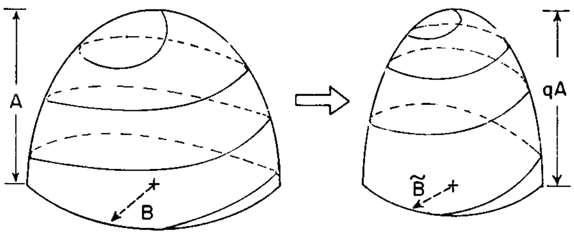
$$EF = 1 - k^3$$

exactly that found for circular fibers. Note that it is not only not possible to attain ejection fractions arbitrarily close to 1.0 for any physiologically reasonable degree of shortening as was true in the cylindrical case, but also that the ejection is wholly independent of the helix parameter Ω.

Ellipse of Revolution

Ejection fraction calculation in the case of helical fibers wrapped on the surface of an ellipse of revolution is more complicated than the preceeding cases in which the ejection fraction could be calculated directly from the given geometric parameters A and B, the fiber parameter Ω and the shortening parameters k and q. Indeed, for each such set of parameters it is now necessary to solve a nonlinear equation (equation 3) for the new semi-minor axis $\tilde{B}$, from which the ejection fraction is calculated as $1 - q(\tilde{B}/B)^2$, where $q(0 < q \leqq 1)$ is the ratio of the lengths of the end systolic semi-major axis and the end diastolic semi-major axis.

Let the left ventricle be approximated at end diastole by a hemi-ellipse of revolution with semi-major axis A and semi-minor axis B as in Fig. 5. The length S of a



FIBER LENGTH S

$$S = B\delta E(K) \longrightarrow kS = \tilde{B}\tilde{\delta}E(\tilde{K})$$

$$A \longrightarrow qA$$

EJECTION FRACTION EF

$$EF = 1 - q\left(\frac{\tilde{B}}{B}\right)^2, \text{ WHERE}$$

$$\tilde{B}\tilde{\delta}E(\tilde{K}) = kB\delta E(K)$$

FIGURE 5 Helical fiber contraction: ellipsoidal surface—variable major axis. Uniform shortening of fiber induces semi-minor axis change simultaneously with semi-major axis shortening.

helical fiber (equation 1 *c*) with parameter Ω is given by

$$S = B\delta E(K, \pi/2)$$

where $K^2 = r^2/(1 + r^2)$, $\delta^2 = 1 + r^2$, $r^2 = \Omega^2 + (A/B)^2 - 1$ and E is the complete elliptic integral of the second kind. Let the ventricle be approximated at end-systole by another ellipse of revolution. Then, if the semi-major axis shortens during systole to qA as the fiber shortens to $kS(0 < k < 1)$, the end systolic semi-minor axis is given by $\tilde{B}$, where

$$\tilde{B}\tilde{\delta}E(\tilde{K}, \pi/2) = kB\delta E(K, \pi/2) \tag{3}$$

provided $kS \geqq qA$. $\tilde{K}$ and $\tilde{\delta}$ depend upon qA/B and the fiber parameter Ω: $\tilde{K}^2 = \tilde{r}^2/(1 + \tilde{r}^2)$, $\tilde{\delta}^2 = 1 + \tilde{r}^2$, $\tilde{r}^2 = \Omega^2 + (qA/\tilde{B})^2 - 1$ and E is as before. The nonlinear equation 3 is solved numerically for $\tilde{B}$ from which the ejection fraction is calculated

$$EF = 1 - q(\tilde{B}/B)^2.$$

Ejection fraction vs. end-diastolic geometry for each of several degrees of fiber shortening at fixed q and Ω values are plotted in Figs. 6–8. The results for $q = 1.0$ (no semi-major axis shortening) are shown in Fig. 6. Two additional cases are presented: $q = 0.95$ in Fig. 7 and $q = 0.90$ in Fig. 8. In all cases Ω has arbitrarily been set equal to 0.1. From Fig. 8 it can be seen that even admitting 10% semi-major axis shortening ($q = 0.9$) it is possible to achieve *EFs* in excess of 0.5 with nominal amounts of fiber shorteining ($k = 0.85$) and reasonable end-diastolic geometry ($B/A = 0.5$). This latter result is still valid for Ω values approaching 1.0. The significance of Ω will be discussed in the section *Fiber Pitch*.

Now, if the end systolic fiber length kS equals the end systolic semi-major axis

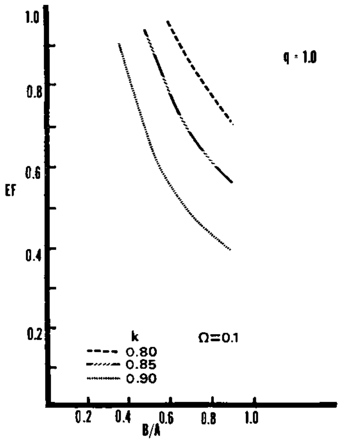


FIGURE 6 Ejection fraction vs. geometry: fixed fiber parameter—ellipse of revolution with fixed major axis ($q = 1.0$). The fiber parameter Ω has arbitrarily been set at $\Omega = 0.1$ and ejection fraction curves depicted for several values of the fiber shortening parameter, $k = 0.80$, 0.85, and 0.90.

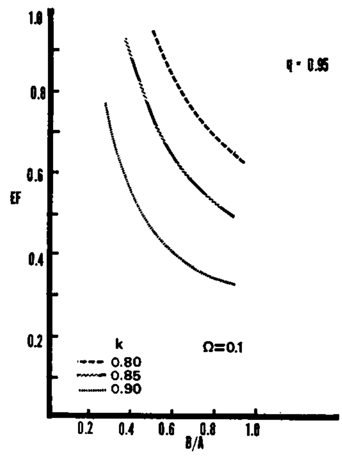


FIGURE 7 Ejection fraction vs. geometry: fixed fiber parameter—ellipse of revolution with variable major axis (q = 0.95). Ejection fraction curves are depicted for several values of the fiber shortening parameter, k = 0.80, 0.85, and 0.90 with a fixed fiber parameter, Ω = 0.1.

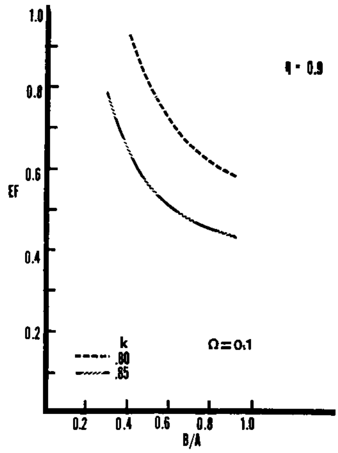


FIGURE 8 Ejection fraction vs. geometry: fixed fiber parameter—ellipse of revolution with variable major axis (q = 0.9). Ejection fraction curves are depicted for several values of the fiber shortening parameter, k = 0.80 and 0.85, with a fixed fiber parameter, Ω = 0.1.

qA, the end systolic semi-minor axis $\bar{B}$ = 0 and ejection is complete. Thus if $S = (q/k)A$ for some Ω the resulting EF = 1.0. It can rigourously be proved that for fixed B/A and q/k ratios such an Ω always exists provided

$$E(\sqrt{1-(B/A)^2}, \pi/2) \leqq q/k$$

where E, as before, is the complete elliptic integral of the second kind.

It then follows that if the endocardial surface of the left ventricle may be reasonably approximated by a hemi-ellipse of revolution at end diastole and end systole it is possible to achieve significant EFs in many cases arbitrarily close to 1.0.

It is not essential that the assumed reference figure be an ellipse of revolution. We may admit two different semi-minor axes such as one usually finds in biplane angiocardiography (7) and consider surfaces which are generalized ellipsoids. Quantitative

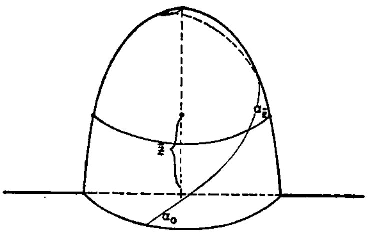


FIGURE 9 Pitch angle definition. The angle $\alpha_{\bar{z}}$ formed by the fiber and the circumference at height $\bar{z}$ is the pitch angle at that z value and is dependent upon z.

changes in the results will occur, of course, but the same qualitative dependence of *EF* on geometry, fiber parameter, semi-major axis shortening, and fiber shortening is found.

Fiber Pitch—Ellipse of Revolution

As is clear from the parametric equation (1 *c*), the helix parameter, Ω, is related to the number of turns the helical fiber makes as it winds down the surface of an ellipse of revolution (e.g., $\Omega = 4.0$ corresponds to one complete revolution). It should be intuitively obvious that the pitch angle α, that is the angle of intersection of the fiber and any circumference (see Fig. 9), depends upon Ω, B/A, and z, and that "small" values of Ω correspond to pitch angles near $\pi/2$ radians while "large" Ω values correspond to pitch angles closer to 0 radians, i.e., nearly circumferential. Specifically, the pitch angle α_0 at the equator ($z = 0$) is determined from:

$$\cos^2 \alpha_0 = \frac{\Omega^2}{[\Omega^2 + (A/B)^2]}. \tag{4}$$

If any other circumference is taken corresponding to some $z = \bar{z}$ then

$$\cos^2 \alpha_{\bar{z}} = \frac{\Omega^2(A^2 - \bar{z}^2)}{[\Omega^2 + (A/B)^2][A^2 - \bar{z}^2] + \bar{z}^2}.$$

DISCUSSION

A mathematical model for the description of helically arranged fibers in the human left ventricle has been presented. Ejection fractions have been calculated for a simple class of helical fibers wound on cylindrical, spherical and ellipsoidal reference ventricles. Normal ejection fractions have been found to be consistent with the concept of helically arranged fibers for a cylinder or ellipse of revolution even undergoing apex to base changes. In contrast, circularly arranged, or constrictor, fibers are inadequate to explain ejection fractions in excess of 0.5.

It should be noted that it has been necessary to piece together a picture of the ventricle by appealing to data from three different animals: the studies of the mechanical properties of isolated papillary muscle come almost exclusively from cats, the studies of fiber orientation have been done on canine hearts and the dynamic geometry of the heart has been most extensively determined from biplane angiocardiograms of human subjects.

In some cases fiber orientation, geometry and *EF* have been studied in the same animals. Ross et al. (8) have reported end diastolic to end systolic dimensional changes in a series of matched canine ventricles. An average *EF* of 0.60 and a q of 0.954 was determined in their study of these ventricles with an average B/A ratio of 0.61. Streeter (oral communication, March 1969) for the same ventricles reports an average endocardial pitch angle as measured at the base of $\alpha_0 = 52.4$ degrees. This corresponds, by equation 4, to $\Omega = 1.27$ if we assume that the endocardial surface of the ventricle is adequately approximated by a hemi-ellipse of revolution with semi-major axis, A, and semi-minor axis, B. Equation 3 may now be solved for $\tilde{B}$ as a function of k from which the ejection fraction may be calculated as a function of k as, for example, in the table below:

k	Ejection fraction ($\Omega = 1.27, B/A = 0.61$)
0.75	0.746
0.80	0.612
0.85	0.437

Thus, the observations of Ross et al. and Streeter would be predicted by approximately a 20% fiber shortening ($k = 0.8$) which is borderline physiologic.

Some care should be taken in attempting to attach any significance to fiber orientation measurements at the endocardial (or epicardial) surface. As pointed out by Streeter et al. (6), measurements at these bounding surfaces are particularly difficult. Indeed the average surface angle values measured are felt to be consistently less than actual. In terms of the helical fiber model discussed here, this would imply by equation 4 a decrease in Ω, which, in turn by equation 3, would predict even larger ejection fractions for a fixed geometry parameter, B/A, and fixed shortening parameters k and q.

Only the simplest class of helical fibers have been described here. Reliable data on myocardial fiber orientation, especially in man, is virtually nonexistent. Should such information become available more complicated modelling of the fiber arrangement may be necessary. However, one should keep in mind that *EF* depends continuously, although complexly, upon the fiber orientation. Thus, "small" deviations in actual fiber orientation from that modelled will produce corresponding "small" changes in *EF*. Even larger deviations of actual vs. modelled fibers, while certainly producing significant quantitative changes should still lead to similar qualitative relationships between geometry, fiber orientation and *EF*.

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